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Some Evidence for the Validity of the Noise-Temperature Inequality $\theta \geq T$ in the Relaxation Approximation of the One-Dimensional Electron Transport Problem in High Electric Fields

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The conjecture that "noise" is always smallest in an equilibrium system is made quantitative for a transport problem by identifying "noise" with the noise temperature θ. In equilibrium the external field $F = 0$, and the fluctuation-dissipation theorem gives $\theta = T$, the temperature. In a strong field F the Boltzmann equation in the constant relaxation approximation is used to calculate the drift $u(F, T)$ the diffusion constant $D(F, T)$, and the noise temperature $\theta(F, T)$ for piecewise linear one-dimensional band structures $E(k)$. The validity of the noise inequality $\theta \geq T$ has been shown for a large variety of band parameters and for all fields and temperatures.

KEY WORDS: Transport problem; stationary nonequilibrium state; nonlinear fluctuation phenomena; noise temperature; diffusion temperature; fluctuation-dissipation theorem; hot electron system; Brownian motion.

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There is a physically very plausible conjecture about the noise of a nonequilibrium system which states that noise should become a minimum in an equilibrium system. In any (stationary) nonequilibrium state it should be larger than its equilibrium value.

Here we want to make this conjecture more quantitative for a transport problem in a strong electric field F in one dimension. It will be shown that the noise temperature θ is the quantity which has the property stated in the conjecture. It is defined according to Price[1] by

$$\theta(F, T) = \pi S(F, T)/\mu(F, T) = D(F, T)/\mu(F, T)$$

where $S(F, T)$ is the velocity fluctuation spectrum, $D(F, T)$ is the diffusion constant, and $\mu(F, T)$ is the differential mobility in a field F in the static case $\omega = 0$.

In equilibrium the field F vanishes and the fluctuation-dissipation theorem (FDT) gives $\theta(0, T) = T$, i.e., the noise temperature coincides with the temperature T. The conjecture therefore becomes in quantitative terms

$$\theta(F, T) \geq T \tag{1}$$

as long as $\mu \geq 0$.[2]

It is expected to hold for all band structures $E(k)$ and for all linear collision operators $\hat{C}$, which gives rise to stable stationary solutions of the transport equation

$$\frac{\partial \phi}{\partial t} + F\frac{\partial \phi}{\partial k} = \hat{C}\phi = \int dk' \, (W_{kk'}\phi_{k'} - W_{k'k}\phi_k)$$

$$W_{kk'}h_{k'} = W_{k'k}h_k, \qquad h_k = Ce^{-E_k/T}, \qquad v_k = \partial E_k/\partial k \tag{2}$$

These equations have been used for transport studies in high electric fields (see Conwell[2]). They implicate collision processes of the electrons with another system, e.g., phonons. If this system has very fast internal relaxation processes, it remains (practically) in equilibrium and acts like an energy reservoir with constant temperature T, e.g., the lattice temperature. The phonon equilibrium assumption has been very successfully applied in order to explain the drift curves in real substances (see, for example, Ref. 3).

In this paper the relevant quantities are calculated explicitly in the constant relaxation-time model of the collision operator $\hat{C}\phi = \nu(h - \phi)$, where $\nu = 1/\tau$ is the relaxation frequency. The band structure $E(k)$ is assumed to be piecewise linear. It is uniquely defined by

$$\begin{aligned} &E(k) \quad \text{continuous}, \qquad E(-k) = E(k), \qquad E(0) = 0 \\ &v(k) = v_n \qquad \text{for} \quad k_n < k < k_{n+1} \quad (n = 0, 1, \ldots, N) \\ &\text{where} \quad k_0 = 0 < k_1 < k_2 < \cdots < k_N < k_{N+1} = \infty \end{aligned} \tag{3}$$

[2] In cases with $\mu < 0$ the assumption of a homogeneous stationary system ($q = 0$, $\omega = 0$) is no longer true and θ loses its physical meaning, but is still defined mathematically.

and is given by the band parameter set

$$k = 0, k_1, k_2, \ldots, k_N, \qquad v = v_0, v_1, v_2, \ldots, v_N$$

The drift velocity in the τ-model (see Ref. 4) is

$$u(F, T) = \bar{v} = \overline{v(k + Ft)}^{ht} \tag{4}$$

and the diffusion constant ($D = \pi S$) is

$$D(F, T) = \int_0^\infty dt\, \overline{\Delta v(t) \Delta v(0)} = A - Bu \tag{5}$$

where

$$A = \overline{v(k + Ft + Fs) v(k + Fs)}^{hts}, \qquad B = \overline{v(k + Ft + Fs)}^{hts}$$

and $\overline{(\cdots)}^h = \int dk(\cdots)\, h$, $\overline{(\cdots)}^t = \int_0^\infty dt(\cdots) \nu e^{-\nu t}$, and analogously for $\overline{(\cdots)}^s$.

In order to evaluate u, A, and B for the piecewise linear band structure (PLBS) it is convenient to express B and part of A by u, C, and its derivatives with respect to F and T

$$B = u + F \frac{du}{dF}, \qquad A = I - \frac{T^2}{F} \left(\frac{du}{dT} - \frac{u}{C} \frac{dC}{dT} \right) \tag{6}$$

Then u and I are given by the double integrals ($\alpha = \nu/F$, $\beta = 1/T$),

$$u = C \int_{-\infty}^{\infty} dk\, e^{-\beta E(k) + \alpha k} \alpha \int_k^\infty dk'\, e^{-\alpha k'} v(k') \tag{7}$$

$$I = C \int_{-\infty}^{\infty} dk\, e^{-\beta E(k) + \alpha k} \alpha^2 \int_k^\infty dk'\, e^{-\alpha k'} v(k') E(k') \tag{8}$$

which decay into $(N + 1)(2N + 3)$ ranges of analytical integrands according to the lines of discontinuity at $k, k' = 0, \pm k_1, \pm k_2, \ldots, \pm k_N$. The evaluation is given in the appendix.

It is easy to calculate u and D in some special cases. If the absolute minimum of $E(k)$ occurs for $k = \pm k_m$ ($k_m = 0, k_1, \ldots, k_N$), then the following $F \to 0$, $T \to 0$ limits hold:

$$\begin{aligned}
\lim_{F \to 0} u(F, 0) &= \tfrac{1}{2}[v(k_m + 0) - v(k_m - 0)] \\
\lim_{F \to 0} D(F, 0) &= \tfrac{1}{4}[v(k_m + 0) + v(k_m - 0)]^2 \\
\lim_{T \to 0} u(0, T) &= 0 \\
\lim_{T \to 0} D(0, T) &= -v(k_m + 0) v(k_m - 0)
\end{aligned} \tag{9}$$

and

$$\mu(0, T) \approx (1/T) \lim_{T \to 0} D(0, T)$$

These formulas hold, since for small T and F all electrons are near the energy minimum $E(k_m)$, and u and D are determined by the energy band parameters near $\pm k_m$. Formulas (9) can also be extended to a band structure with more equal or nearly equal energy minima by weighting the contributions with the corresponding Boltzmann factors $\exp[-E(k_m)/T]$.

In the limit of $F \to \infty$ or $T \to \infty$ the following is true:

$$T = \infty: \quad u(F, \infty) = 0, \quad D(F, \infty) = v_N^2 \tag{10}$$
$$F = \infty: \quad u(\infty, T) = v_N, \quad D(\infty, T) = 0$$

This result is again plausible, since for high field or temperature most of the electrons have very high energy and u and D are determined by v_N, the asymptotic velocity. It is remarkable that $F = 0$, $T = 0$ and $F = \infty$, $T = \infty$ values of u and D depend on the order in which the limits are taken.

The drift curves $u(F, T)$ have been published earlier (see Ref. 5) for the band parameters

$$\begin{pmatrix} 0, k_1, \dots, k_N \\ v_0, v_1, \dots, v_N \end{pmatrix}$$

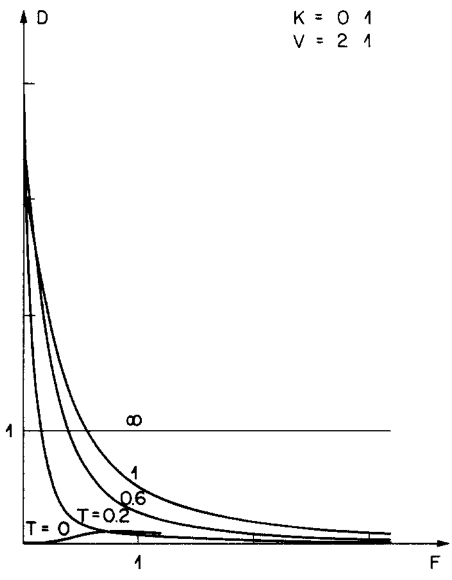

Fig. 1. The diffusion constant $D(F, T)$ as a function of field F with temperature T as a parameter for $\nu = 1$ and the specified band structure $k = 0, 1$; $v = 2, 1$ [for $u(F, T)$ see Figs. 3–7 in Ref. 5].

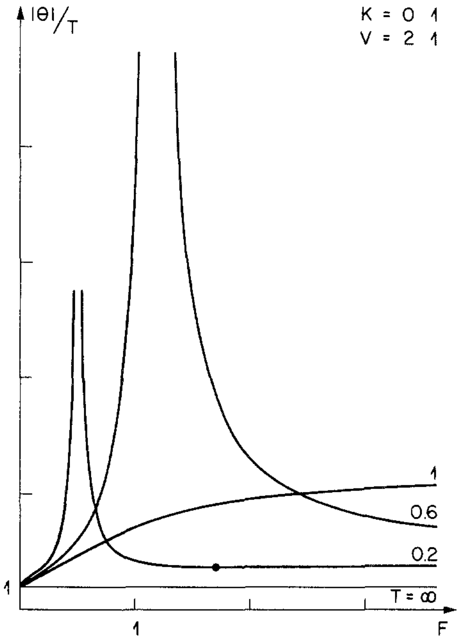

Fig. 2. The magnitude of the relative noise temperature $|\theta(F, T)|/T$ as a function of field F with temperature T as a parameter for $\nu = 1$ and the specified band structure $k = 0, 1$; $v = 2, 1$ [for $\mu(F, T) \lessgtr 0$ see Figs. 3–7 in 5].

for

$$\begin{pmatrix}0, 1\\2, 1\end{pmatrix}, \quad \begin{pmatrix}0, 1\\-2, 1\end{pmatrix}, \quad \begin{pmatrix}0, 1\\0.5, 1\end{pmatrix}, \quad \begin{pmatrix}0, 1\\-0.5, 1\end{pmatrix}, \quad \begin{pmatrix}0, & 0.5, & 1\\2, & -1.9, & 1\end{pmatrix}$$

In Figs. 1–10 the diffusion constant $D(F, T)$ and the magnitude of the relative noise temperature $|\theta(F, T)|/T$ are given as functions of F with T as a parameter for the same band parameters as above. Putting $\nu = 1$ is no essential restriction, since ν enters only in the combinations F/ν and $D\nu$, which are the quantities actually plotted. The detailed structure of the curves depends very much on the parameters chosen. In general, D and $|\theta|/T$ have maxima and minima for finite fields, and $D \to 0$ as $F \to \infty$ [see (10)], whereas $|\theta|/T$ tends to a limit value for $F \to \infty$ (see the appendix). For small fields the discontinuous behavior of (9) becomes evident by comparing the curves for small T with $T = 0$ result.

It can easily be seen that in our cases the conjecture $\theta \geq T$ is everywhere fulfilled.[3] The computer program written has been used for many other

[3] A proof can be given for small fields in the τ-model.

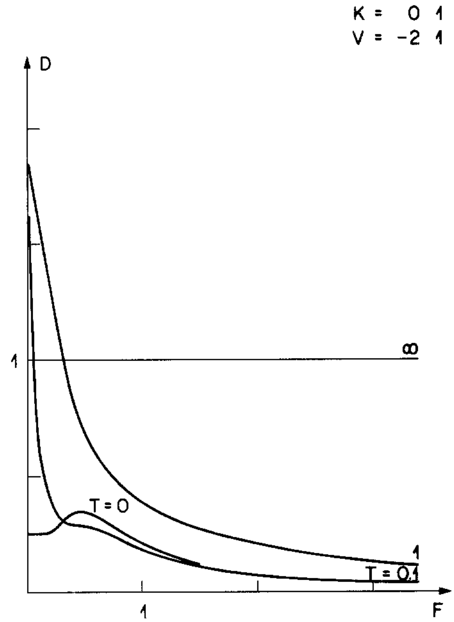

Fig. 3. D vs. F, as in Fig. 1, but for $k = 0, 1$; $v = -2, 1$.

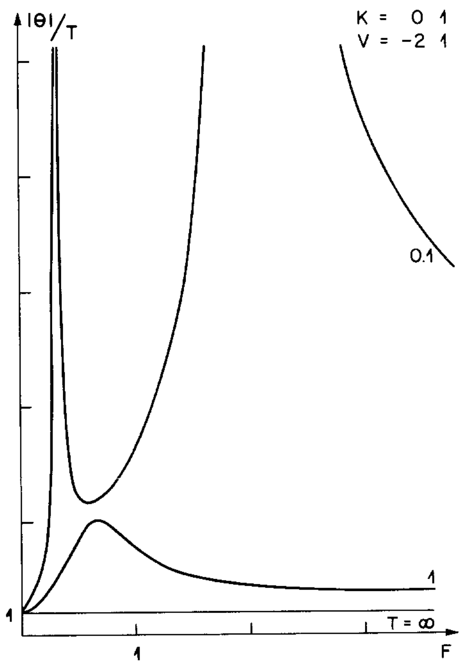

Fig. 4. $|\theta|/T$ vs. F, as in Fig. 2, but for $k = 0, 1$; $v = -2, 1$.

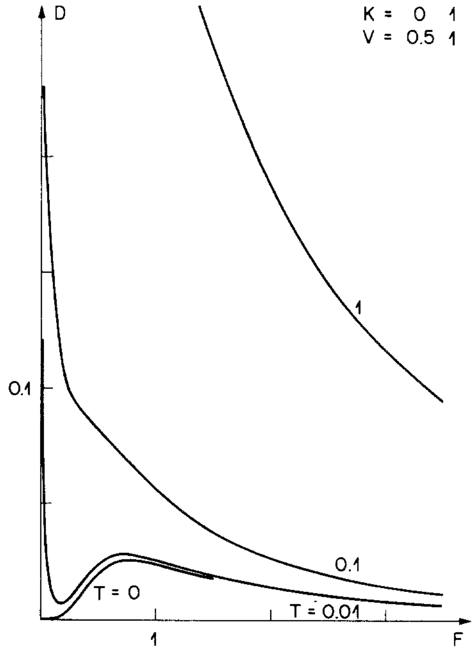

Fig. 5. D vs. F, as in Fig. 1, but for $k = 0, 1$; $v = 0.5, 1$.

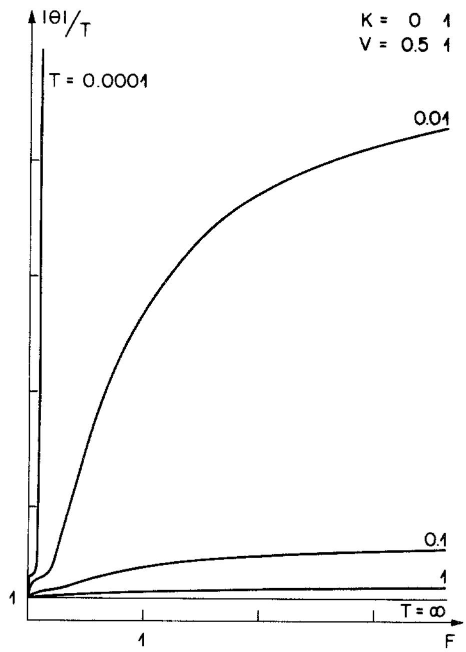

Fig. 6. $|\theta|/T$ vs. F, as in Fig. 2, but for $k = 0, 1$; $v = 0.5, 1$.

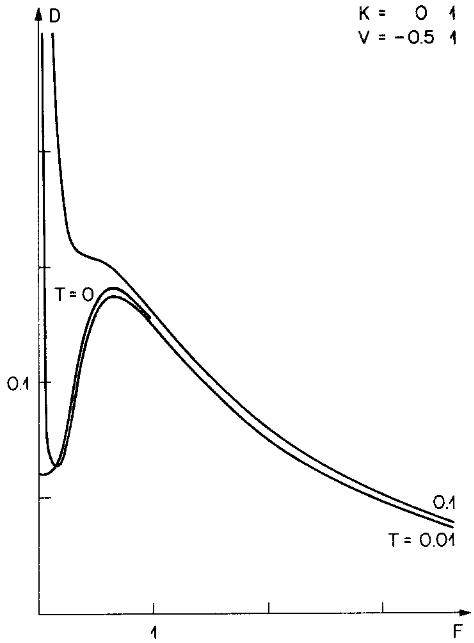

Fig. 7. D vs. F, as in Fig. 1, but for $k = 0, 1$; $v = -0.5, 1$.

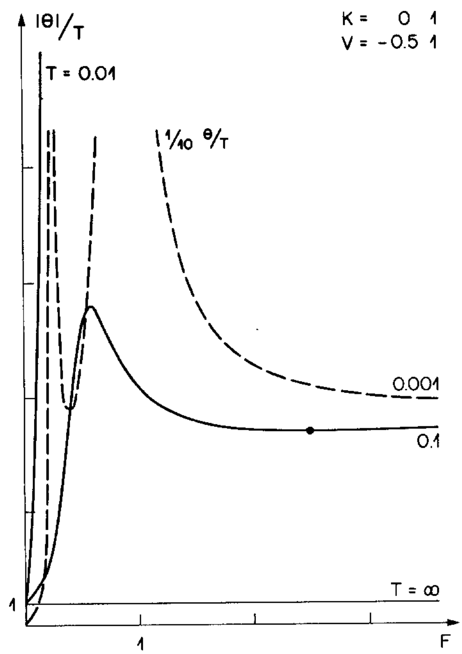

Fig. 8. $|\theta|/T$ vs. F, as in Fig. 2, but for $k = 0, 1$; $v = -0.5, 1$.

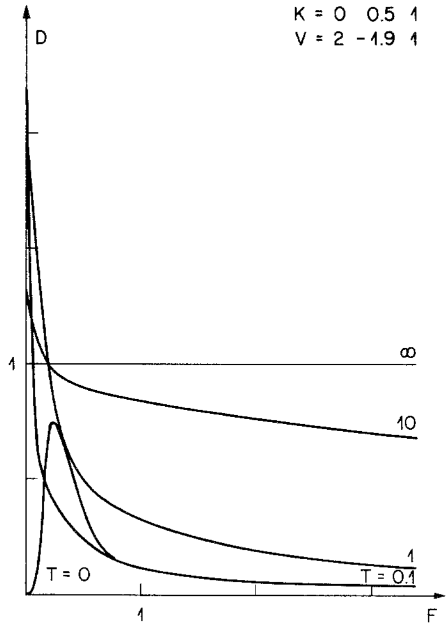

Fig. 9. D vs. F, as in Fig. 1, but for $k = 0$, 0.5, 1; $v = 2, -1.9, 1$.

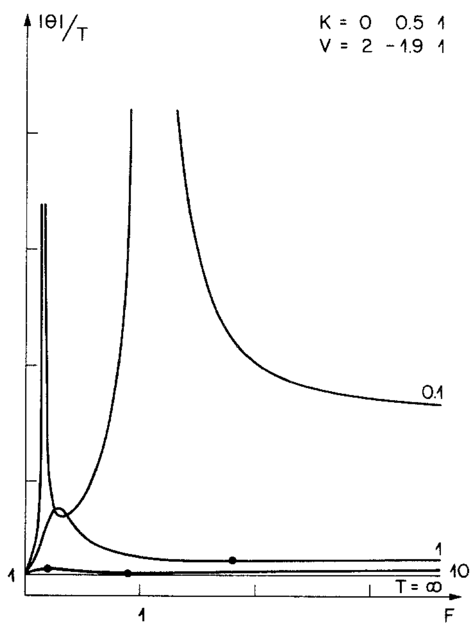

Fig. 10. $|\theta|/T$ vs. F, as in Fig. 2, but for $k = 0$, 0.5, 1; $v = 2, -1.9, 1$.

examples, and parameters have been changed in such a way as to make the θ/T minima (>1) smaller and eventually to get a case with $\theta/T < 1$; however, this could never be accomplished.

We do not consider this as a proof of the conjecture in the relaxation model of the collision operator, but we hope to have shown that there is strong evidence for the validity of the conjecture.

APPENDIX

In order to evaluate (7) and (8) it is convenient to use a vector notation[4]

$$\{k(0),\ldots, k(2N+2)\} = \{-\infty, -k_N, -k_{N-1},\ldots, -k_1, 0, k_1, k_2,\ldots, k_N, \infty\}$$

$$\{v(1),\ldots, v(2N+2)\} = \{-v_N, -v_{N-1},\ldots, -v_0, v_0, v_1,\ldots, v_N\}$$

$$E(n+1) = E(n) + v(n+1)[k(n+1) - k(n)], \qquad E(N+1) = 0$$

$$\frac{1}{C} = \int_{-\infty}^{\infty} dk\, e^{-\beta E} = \sum_{n=0}^{2N+1} \int_{k(n)}^{k(n+1)} dk\, e^{-\beta E} = \sum_{n=0}^{\infty} \frac{e^{-\beta E(n)} - e^{-\beta E(n+1)}}{\beta v(n+1)}$$

$$u = \sum_{n=0}^{2N+1} \sum_{l=n}^{2N+1} u_{nl}$$

$$u_{nl} = \int_{k(n)}^{k(n+1)} dk \int_{k(l)}^{k(l+1)} dk'\,(\cdots) \qquad \text{for} \quad n > l$$

$$u_{nn} = \int_{k(n)}^{k(n+1)} dk \int_{k(n)}^{k(n+1)} dk'\,(\cdots) \qquad \text{for} \quad n = l$$

and analogous definitions for I, I_{nl}, I_{nn},

$$u_{nl} = C\,\frac{e(n) - e(n+1)}{\beta v(n+1) - \alpha}\, v(l+1)(e^{-\alpha k(l)} - e^{-\alpha k(l-1)})$$

$$u_{nn} = Cv(n+1)\left(\frac{e^{-\beta E(n)} - e^{-\beta E(n+1)}}{\beta v(n+1)} - e^{-\alpha k(n+1)}\,\frac{e(n) - e(n+1)}{\beta v(n+1) - \alpha}\right)$$

$$I_{nl} = C\,\frac{e(n) - e(n+1)}{\beta v(n+1) - \alpha}\, v(l+1)[v(l+1)(e^{-\alpha k(l)} - e^{-\alpha k(l+1)})$$

$$+ \alpha[E(l)e^{-\alpha k(l)} - E(l+1)e^{-\alpha k(l+1)}\}$$

[4] In an APL program vectors can be handled in a very efficient way.

$$I_{nn} = -C\frac{e(n) - e(n+1)}{\beta v(n+1) - \alpha} v(n+1)[\alpha E(n+1) + v(n+1)]e^{-\alpha k(n+1)}$$
$$+ C\frac{e^{-\beta E(n)} - e^{-\beta E(n+1)}}{\beta}\{\alpha E(n) + v(n+1)[1 - \alpha k(n)]\}$$
$$+ C\frac{\alpha}{\beta^2}\{e^{-\beta E(n)}[1 + \alpha k(n)] - e^{-\beta E(n+1)}[1 + \alpha k(n+1)]\}$$

where $e(n) = e^{-\beta E(n) + \alpha k(n)}$.

For periodic band structures $E(k + p) = E(k)$ the limit $F \to \infty$ gives

$$\frac{|\theta(\infty, T)|}{T} = \frac{\tilde{E}^2 - 2\tilde{E}\overline{E}^h + \overline{E^{2h}}}{2(\tilde{E} - \overline{E}^h)}$$

where $(\tilde{\cdots}) = (1/p)\int_{(p)} dk\,(\cdots)$ is the average over the period. For $T \to \infty$

$$|\theta(\infty, T)|/T \gtrsim 1$$

A similar result holds for the nonperiodic band structures ($p = \infty$), as, for example, for the PLBS, but it depends again on the order of the limits $T \to \infty$ and $F \to \infty$.

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